Waves & Coffee – Impedance matching

Mathematically speaking, why a Z_0 of $50\ \Omega$?

By Stephen W2WF

Hams are by and large familiar with the concept of 'impedance matching'. That's when your coax cable needs to present a 50-ohm resistance to your transmitter to make it happy. But why is that?

Well, according to the ARRL Antenna book, a study was published shortly after World War II by the MIT Radiation Lab. The author of Volume 9 in the series, George L. Ragan concluded: "Obvious economy both in test equipment and in design work can be achieved if a single impedance can be chosen as a compromise standard. It has been found convenient to adopt 50 ohms as an impedance level offering a satisfactory compromise."

Oh, great. Not much more to say about the choice of 50 ohms for the 'characteristic impedance' of coax cables. It seems that it was a simple engineering compromise.

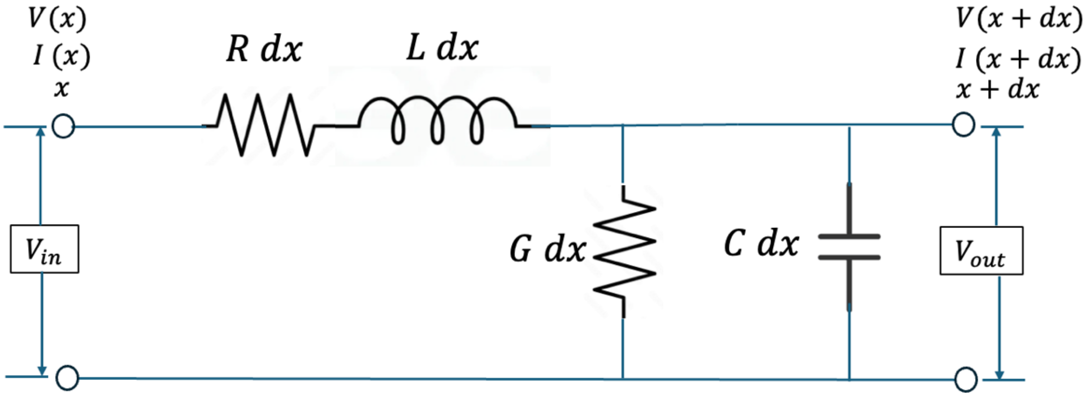
Of course, Hams being Hams may still want to know the *reason* behind this number. For example, what's the mathematics behind impedance matching? What formula do we use for this 'characteristic impedance' of a coax cable, or any cable for that matter. If there is a formula, where does it come from?

It turns out that impedance matching of a uniform transmission line requires us to start with a result that we derived previously in this column: Oliver Heaviside's Telegrapher's Equations. Starting from there, we can work out the math to get an expression for the 'characteristic impedance', or 'surge impedance', of a uniform transmission line. Note that we could also call it the 'input impedance' if we assume its length to be infinite, but we digress.

So, let's do the physics! Let's derive the formula for the Characteristic Impedance, Z_o, of a uniform transmission line! We can then plug-n-chug some values for a popular coax cable, say RG-8X, and see if we get 50 ohms for characteristic impedance.

As mentioned, our approach here will be to build on work done in the previous columns in this series. So, in case we need to review where these equations come from, I'd like to point you to Part I and Part II of the telegrapher's equations in this series.

In those previous columns we looked at Oliver Heaviside's analysis of the transmission line. Recall that we looked at his model for a transmission line which looks like this:



Where,

$R = resistance\ per\ unit\ length$

$L = Inductance\ per\ unit\ length$

$C = capacitance\ per\ unit\ length$

$G = conductance\ per\ unit\ length$

$\mathcal{G} = conductance$

$\mathcal{R} = resistance$

After following Heaviside's amazing work, we found that the first of the "Telegrapher's Equations" is the following:

$$\frac{\partial V(x,t)}{\partial x} = -R\, I(x,t) - L\frac{\partial I(x,t)}{\partial t} \tag{1}$$

In the next step, we will solve this equation... but don't panic! We'll tackle the problem methodically and explain each step.

The way we solve equations that contain derivatives is, believe it or not, by guessing a solution and seeing if our guess works. Now, this may seem arbitrary, but we'll be careful about what we use as a guess; furthermore, our experience tells us that using the exponential function $f(x) = e^x$ is a good guess.

So, we guess a solution for Voltage and Current using the exponential function as follows:

$$V(x,t) = V_o e^{-(\gamma x - iwt)} \qquad I(x,t) = I_o e^{-(\gamma x - iwt)}$$

Where gamma (γ) and little omega (ω) are constants, in other words, they don't depend on x or t. Of course, $i = \sqrt{-1}$ which is the standard math/physics definition. We then plug in these two expressions into equation (1) and get,

$$-\gamma V_o e^{-(\gamma x - iwt)} = -R\, I_o e^{-(\gamma x - iwt)} - i\omega L I_o e^{-(\gamma x - iwt)}$$

Simplifying the right-hand side gives,

$$-\gamma\, V_o e^{-(\gamma x - iwt)} = (-R - i\omega L)\, I_o e^{-(\gamma x - iwt)}$$

Multiplying both sides by -1 and isolating V_o and I_o to the left-hand side, we get this intermediate result:

$$\frac{V_o}{I_o} = \frac{(R + i\omega L)}{\gamma} \tag{2}$$

Similarly we start with the second telegrapher equation that we previously derived and proceed exactly as above.

$$\frac{\partial I(x,t)}{\partial x} = -\,G\, V(x,t) - C\,\frac{\partial V(x,t)}{\partial t} \tag{3}$$

First, we guess a solution for Voltage and Current as before:

$$V(x,t) = V_o e^{-(\gamma x - iwt)} \qquad I(x,t) = I_o e^{-(\gamma x - iwt)}$$

Plugging and chugging into our equation (3), we get the following,

$$-\gamma I_o e^{-(\gamma x - iwt)} = -G\, V_o e^{-(\gamma x - iwt)} - i\omega C V_o e^{-(\gamma x - iwt)}$$

Simplifying the right-hand side gives,

$$-\gamma\, I_o e^{-(\gamma x - iwt)} = (-G - i\omega C)\, V_o e^{-(\gamma x - iwt)}$$

Multiplying both sides by -1 and isolating V_o and I_o on the left-hand side,

$$\frac{V_o}{I_o} = \frac{\gamma}{(G + i\omega C)} \tag{4}$$

The last step is to Solve (4) for γ and to do that we multiply both sides by $(G + i\omega C)$, to get,

$$\frac{V_o}{I_o}(G + i\omega C) = \gamma$$

We put the gamma on the left-hand side to make things neater,

$$\gamma = \frac{V_o}{I_o}(G + i\omega C)$$

and plug that result into (2),

$$\frac{V_o}{I_o} = \frac{(R + i\omega L)}{\frac{V_o}{I_o}(G + i\omega C)}$$

Multiplying by $\frac{V_o}{I_o}$ gives us,

$$\left(\frac{V_o}{I_o}\right)\left(\frac{V_o}{I_o}\right) = \frac{(R + i\omega L)}{(G + i\omega C)}$$

$$\left(\frac{V_o}{I_o}\right)^2 = \frac{(R + i\omega L)}{(G + i\omega C)}$$

Finally, taking the square root of both sides, gives,

$$\frac{V_o}{I_o} = \sqrt{\frac{R + \ i\omega L}{G + i\omega C}}$$

This quantity has units of resistance and so we call it the impedance of our transmission line. If we look at it closely, we notice that it's specific to our transmission line. So it really is a "characteristic impedance" which is what it's called in the literature, where it's given the symbol Z_o.

$$\boxed{Z_o = \sqrt{\frac{R + \ i\omega L}{G + i\omega C}}}$$

In passing we note that the square root will have positive and negative values for Z_o corresponding to the forward and the backward propagating waves in the standing wave pattern.

Also, note that if the transmission line is lossless, we'd have $R = G = 0$ and our characteristic impedance reduces to:

$$\boxed{Z_o = \sqrt{\frac{\cancel{R} + \ \cancel{i\omega}L}{\cancel{G} + \cancel{i\omega}C}}}$$

$$Z_o = \sqrt{\frac{L}{C}}$$

Ok, as the last step, we'll put in some values in our equation above to see if it agrees with what we'd expect.

Per the Belden data sheet in the reference section, for RG8-X coax cable, L = .065 μH and C = 24.8 pF. So, plugging those values in,

$$Z_o = \sqrt{\frac{L}{C}} = \sqrt{\frac{0.065\ x\ 10^{-6}}{23.8\ x\ 10^{-12}}} = \boxed{51.19\ \Omega}$$

Which is the result that we expected! Well, plus or minus 2% tolerance.

Bibliography

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