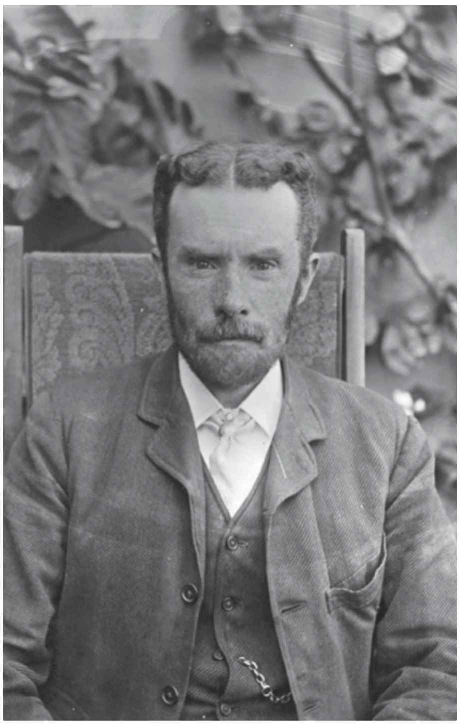
Waves & Coffee –

Telegrapher's Equations - Part I

Oliver Heaviside's work that enabled telegraphy

By Stephen W2WF



Oliver Heaviside was a fascinating scientist who lived in the 1800's. He was a driven and mostly self-taught mathematician and electrical engineer. One of his accomplishments was the synthesis of Maxwell's equations from the original 20 equations down to 4 using Vector Calculus. So, if you've seen the familiar 4 "Maxwell's Equations", then you may be keen to know that writing them in that compact and elegant form is due to Oliver Heaviside.

Another seminal work attributed to Mr. Heaviside would be the derivation of the so called "Telegrapher's Equations". These result from his rigorous analysis of the physics of the transmission line using the concept of lumped parameters.

The motivation for this was the very practical problem of the feasibility of economical transatlantic communication cables. This problem had been tackled by the well-known scientist William Thomson, Lord Kelvin. He'd come up with an equation that considered the resistance and the capacitance of a transmission line. However, his analysis resulted in an equation called "The Diffusion equation". This was not good since the model showed that the signals from Europe would *diffuse* or *disperse* before they reached North America. Lord Kelvin's model showed that transatlantic cables could be viable, if you used *extremely* conducting cables. Such cables would simply be too expensive.

So up to this point in our story, the theoretical analysis didn't make transatlantic cable communication sound promising. The project was at a standstill.

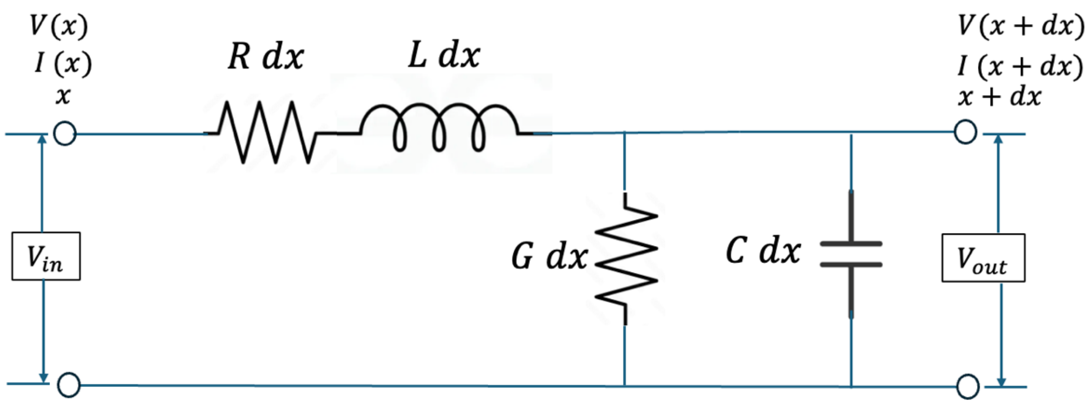
Enter Oliver Heaviside and his thorough study of the problem. His analysis came up with a slightly different model for the transmission line since it included an additional term: the inductance of the transmission line. Thanks to this modification, Heaviside's model no longer resulted in a diffusion equation but rather a "Wave equation". That in turn meant that the transatlantic cables would be economically viable even if you didn't have expensive conducting cables. This made transatlantic communication via undersea cables viable!

First, we define what "lumped parameters" are and how they apply to the circuit analysis of our problem. Lumped parameters are simple circuit elements which substitute for much more complex objects that may radiate, dissipate heat, and in general exhibit non-linear behaviour. When we consider circuit elements as lumped parameters, we ensure that they're driven in their linear regime and neglect the other interesting physical aspects just mentioned. We concentrate on them as being essentially idealized components that can be used in circuit diagrams as building blocks.

The two telegrapher's equations, as well as the transmission line wave equation, emerge from such a lumped parameter circuit analysis.

Our strategy will be to first model our transmission line by drawing it as a circuit diagram and then apply Kirchoff's Voltage Law (KVL). The second part of our strategy will be to apply Kirchoff's Current Law (KCL) to our circuit.

Now, let's do physics!



Where,

$R = resistance\ per\ unit\ length$

$L = Inductance\ per\ unit\ length$

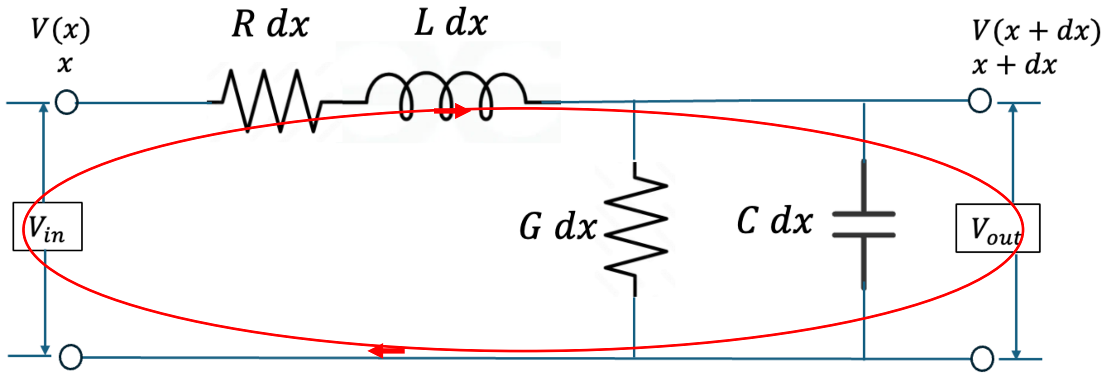
$C = capacitance\ per\ unit\ length$

$G = conductance\ per\ unit\ length$

$\mathcal{G} = conductance$

$\mathcal{R} = resistance$

The first equation comes from applying KVL around this loop:



KVL says that the sum of voltages and voltage drops around a closed loop is zero. So, if we start at $V(x,t)$, our sum looks like this:

$$V(x,t) - R\,dx\ - L\,dx - \ V(x+dx,t) = 0$$

But there's something not right in this expression since our units for $R\,dx$ and $L\,dx$ are not right. We need to work in units of voltage to be able to make the units consistent.

Ok, for the first term we'll use Ohm's law $V = I\mathcal{R}$ and for the second term we'll use the definition of inductance in terms of a time-varying current: $V = \ L\,dx\ \frac{dI}{dt}$.

$$V(x) - R\,I\,dx - L\,dx\frac{dI}{dt} - V(x+dx) = 0$$

To remind ourselves that our $V(x)$ and $I(x)$ variables can vary with time, as well as distance, we replace the total derivative with a partial derivative,

$$V(x,t) - R\,I(x,t)\,dx - L\,dx\frac{\partial I(x,t)}{\partial t} - V(x+dx,t) = 0$$

Rearranging our expression to have the total voltage drop in the interval dx, our equation becomes:

$$V(x+dx,t) - V(x,t) = \ -R\,I(x,t)\,dx - L\,dx\frac{\partial I(x,t)}{\partial t}$$

Dividing both sides of the equation by dx and then taking the limit as $dx\ \to 0$,

$$\lim_{dx\to 0}\frac{V(x+dx,t) - V(x,t)}{dx} = \ -R\,I(x,t)\ - L\frac{\partial I(x,t)}{\partial t}$$

This gives the first of the "Telegrapher's Equations":

$$\frac{\partial V(x,t)}{\partial x} = \ -R\,I(x,t)\ - L\frac{\partial I(x,t)}{\partial t} \qquad (1)$$

This equation shows that the voltage drop along the transmission line segment is related to the resistive and inductive properties of the line.

In the second half of this two-part article, we'll use Kirchoff's Current Law (KCL) to derive the 2nd Telegrapher equation as well as the wave equation for this system.

Bibliography

Donaghy-Spargo C. 2018 On Heaviside's contributions to transmission line theory: waves, diffusion and energy flux. Phil. Trans. R. Soc. A 376: 20170457. http://dx.doi.org/10.1098/rsta.2017.0457

Maxwell's Equations – Wikipedia - https://en.wikipedia.org/wiki/Maxwell's_equations

Telegrapher's Equations -Wikipedia- https://en.wikipedia.org/wiki/Telegrapher%27s_equations

Heaviside O. 1892 Electrical papers, vol. II, p. 82. London, UK: Macmillan and Co

Heaviside O. 1892 Electrical papers, vol. II, pp. 137–141. London, UK: Macmillan and Co.