Waves & Coffee –
The Skin Effect

What is ‘skin depth’ all about?

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Consider three seemingly unrelated scenarios.

Nowadays, we can go to Amazon.com and order a “Faraday cage envelope” to place our cell phones inside. The Faraday cage envelope keeps the E&M waves from going out, or coming into, the item that we have inside. This is usually done for privacy reasons and works well. But *why* does this work?

Similarly, when we watch some reputable YouTube videos, we see that in a copper wire that carries current, the current density tends to concentrate on the surface, or skin, of the conductor. If that is indeed what happens, why does it happen?

Lastly, we’ve all watched movies or read books in which submarines go to shallow depths to communicate with headquarters using something called ‘VLF’ or ‘Very Low Frequency’ radio waves. Why don’t they use HF, VHF, UHF, or even microwaves to communicate below the water? And if they indeed come up to a shallow depth for comms, why do they do that?

Are these three different phenomena related?

The short answer is yes, it turns out that these are examples of a physical phenomenon called ‘The Skin Effect’. The name itself is descriptive in that it refers to the current density of a conductor tending to concentrate on the surface of the conductors; furthermore, the skin effect has a dependence on the frequency of the E&M wave being considered.

Since we’re dealing with E&M waves, let’s begin with Maxwell’s equations and see if we can come up with an expression for ‘skin depth’ which depends on our wave’s frequency. As before, we’ll go step by step and simplify the concepts down to the essential physics.

We will be following the excellent derivation found in the Manchester Physics Series book mentioned in the reference section.

We start with the following two Maxwell equations: the one relating the rotation of the E field to the time rate of change of the B field

$$\boldsymbol{\nabla x \vec{E}} = -\frac{\partial \vec{\boldsymbol{B}}}{\partial t} \qquad (1)$$

And the one relating the rotation of the B field to the current and the changing E field,

$$\boldsymbol{\nabla x}\,\vec{\boldsymbol{B}} = \mu\vec{\boldsymbol{J}} + \mu\epsilon\frac{\partial\vec{\boldsymbol{E}}}{\partial t} \tag{2}$$

Considering only materials where the current density is proportional to the conductivity times the E field, i.e. Ohm's law,

$$\vec{\boldsymbol{J}} = \sigma\vec{\boldsymbol{E}} \tag{3}$$

We insert equation (3), ohm's law, into equation (2) and get,

$$\boldsymbol{\nabla x}\,\vec{\boldsymbol{B}} = \mu\sigma\vec{\boldsymbol{E}} + \mu\epsilon\frac{\partial\vec{\boldsymbol{E}}}{\partial t} \tag{4}$$

Now we use the following vector identity which can be found in mathematical tables such as the one mentioned in the reference section,

$$\boldsymbol{\nabla x}\,\boldsymbol{\nabla x}\,\vec{\boldsymbol{E}} = -\nabla^2\vec{\boldsymbol{E}} + \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\vec{\boldsymbol{E}})$$

And apply it to equation (1), after we operate with the $\boldsymbol{\nabla x}$ or curl operator on both sides,

$$\boldsymbol{\nabla x}\,\boldsymbol{\nabla x}\,\vec{\boldsymbol{E}} = -\nabla^2\vec{\boldsymbol{E}} + \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\vec{\boldsymbol{E}}\right) = \boldsymbol{\nabla x}\,(-\frac{\partial\vec{\boldsymbol{B}}}{\partial t})$$

But in a conductive medium with no charge accumulation, $\boldsymbol{\nabla}\cdot\vec{\boldsymbol{E}} = 0$, so we can simplify a bit,

$$\boldsymbol{\nabla x}\,\boldsymbol{\nabla x}\,\vec{\boldsymbol{E}} = -\nabla^2\vec{\boldsymbol{E}} + \boldsymbol{\nabla}\left(\cancel{\boldsymbol{\nabla}\cdot\vec{\boldsymbol{E}}}\right) = \boldsymbol{\nabla x}\,(-\frac{\partial\vec{\boldsymbol{B}}}{\partial t})$$

multiplying both sides by -1,

$$-1\,x\,(-\nabla^2\vec{\boldsymbol{E}}) = -1\,x\,(-\boldsymbol{\nabla x}\,\frac{\partial\vec{\boldsymbol{B}}}{\partial t})$$

We get,

$$\nabla^2\vec{\boldsymbol{E}} = \boldsymbol{\nabla x}\,\frac{\partial\vec{\boldsymbol{B}}}{\partial t}$$

Now we take the curl on the right-hand side inside the derivative,

$$\nabla^2\vec{\boldsymbol{E}} = \frac{\partial(\boldsymbol{\nabla x}\,\vec{\boldsymbol{B}})}{\partial t} \tag{5}$$

Next, we insert equation (4) into equation (5),

$$\nabla^2\vec{\boldsymbol{E}} = \frac{\partial(\,\mu\sigma\vec{\boldsymbol{E}} + \mu\epsilon\frac{\partial\vec{\boldsymbol{E}}}{\partial t})}{\partial t}$$

And get,

$$\nabla^2\vec{\boldsymbol{E}} = \mu\sigma\frac{\partial\vec{\boldsymbol{E}}}{\partial t} + \mu\epsilon\frac{\partial^2\vec{\boldsymbol{E}}}{\partial t^2}$$

We note that we're only considering the z direction, perpendicular to the metal surface, so we can replace the left-hand side of the equation with a simple derivative in the z direction,

$$\frac{\partial^2 \vec{\boldsymbol{E}}}{\partial z^2} = \mu\sigma \frac{\partial \vec{\boldsymbol{E}}}{\partial t} + \mu\epsilon \frac{\partial^2 \vec{\boldsymbol{E}}}{\partial t^2}$$

Next, we neglect the 2nd term on the right-hand side of this equation. We do this since it's the displacement current term and for the frequencies that we'll be working with it's very small.

$$\frac{\partial^2 \vec{\boldsymbol{E}}}{\partial z^2} = \mu\sigma \frac{\partial \vec{\boldsymbol{E}}}{\partial t} + \cancel{\mu\epsilon \frac{\partial^2 \vec{\boldsymbol{E}}}{\partial t^2}}$$

Our equation finally reduces to this,

$$\boxed{\frac{\partial^2 \vec{\boldsymbol{E}}}{\partial z^2} = \mu\sigma \frac{\partial \vec{\boldsymbol{E}}}{\partial t}}$$

As with other equations of this type, we solve it using a technique called... well, it's called "guessing at a solution". Of course, we use a smart guess based on our experience!

Let's try this solution, and note that here $j = \sqrt{-1}$ per the electrical engineering convention,

$$\vec{\boldsymbol{E}} = \overrightarrow{\boldsymbol{E_o}}\, exp[j(\omega t - \beta z)]\, exp(-\alpha z)$$

Ok, so let's evaluate our equation above and see if this proposed solution works.

$$\mu\sigma \frac{\partial \vec{\boldsymbol{E}}}{\partial t} = \mu\sigma(j\omega)\overrightarrow{\boldsymbol{E_o}}\, exp[j(\omega t - \beta z)]\, exp(-\alpha z)$$
$$\frac{\partial^2 \vec{\boldsymbol{E}}}{\partial z^2} = (\alpha + j\beta)^2\, \overrightarrow{\boldsymbol{E_o}}\, exp[j(\omega t - \beta z)]\, exp(-\alpha z)$$

Now, we set both sides equal and cancel the corresponding terms to obtain,

$$(\alpha + j\beta)^2 = \mu\sigma(j\omega)$$

Multiplying out the terms on the left,

$$\alpha^2 - \beta^2 + 2j\alpha\beta = \mu\sigma(j\omega)$$

Now we consider the case in which $\alpha = \beta$ and terms nicely cancel,

$$\cancel{\alpha^2 - \beta^2} + 2j\alpha\beta = \mu\sigma(j\omega)$$

$$2\cancel{j}\alpha^2 = \mu\sigma(\cancel{j}\omega)$$

$$\alpha^2 = \frac{\mu\sigma\omega}{2}$$

$$\alpha = \sqrt{\frac{\mu\sigma\omega}{2}}$$

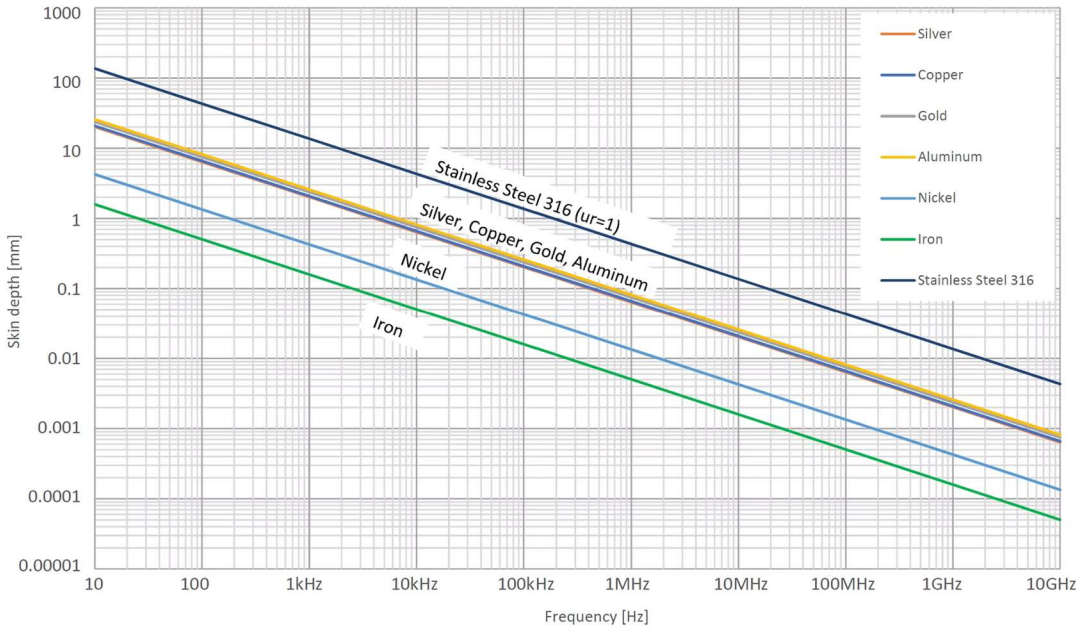
And we note that the inverse of α has units of length. So, the wave attenuates by a factor of 1/e if it travels a distance $1/\alpha$. We call this distance the 'skin depth' and denote it by the Greek letter delta δ:

$$\delta \equiv \frac{1}{\alpha} = \sqrt{\frac{2}{\mu\sigma\omega}}$$

This is the 'skin depth' equation. We note that this effect goes as $\sim\sqrt{\frac{1}{\omega}} = \sqrt{\frac{1}{2\pi f}}$. So, the higher the frequency f, the smaller the skin depth parameter δ gets.

Trying out our formula for salt water, we have $\sigma = 3.3\, S/m$, $\mu \cong \mu_o = 4\pi\, x\, 10^{-7} H/m$, $f = 80\, Hz$, which gives $\delta = 31\, m$ which in imperial units is $\delta = 101.7\, ft.$ Which is why subs must come up shallow.

Finally, we confirm that for metals like copper, silver, aluminum, iron, and stainless steel the skin depth is very close to the surface of the conductor; furthermore, it gets smaller (fast!) as the frequency increases (note the logarithmic scale):



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https://en.wikipedia.org/wiki/Vector_calculus_identities

Website discussing skin effect in various materials https://www.academyofemc.com/skin-effect