Waves & Coffee –
The coaxial cable

A mathematical look at the COAX cable
By Stephen W2WF

The coaxial cable, or coax, is well known to amateur radio operators around the world. It is a versatile and sturdy transmission line that hams typically use from their transceiver out to the feed point of their antenna. It is so widely used that, at times, we may take this amazing bit of hardware for granted.

We previously looked at the analysis of the transmission line undertaken by Oliver Heaviside. His was the amazing story of a self-taught physicist and engineer from England in the 19th century who went against conventional wisdom and helped make the transatlantic cable a reality. His analysis used "lumped parameters" which are typically used in electronic circuits. These lumped parameters are inductors, resistors, capacitors that are taken as ideal components. In other words, we neglected thermal effects, field effects, geometry and assumed that our lumped parameters were *well behaved, linear, idealized components*.

We saw how Heaviside's analysis of the transmission line resulted in the concept of characteristic impedance Z_o. We concluded by computing Z_o of a coax cable to be equal to 50 ohms.

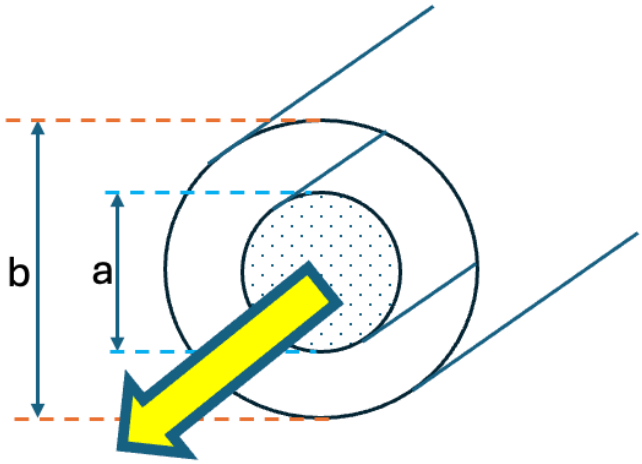
In this column we'll take a different approach. In particular we won't use lumped parameters but rather attack the coax transmission line problem starting from $\vec{E}$ and $\vec{B}$ fields. When we talk about $\vec{E}$ and $\vec{B}$ fields, the equations that govern them are, of course, Maxwell's equations.

"Field equation analysis" may sound like a very difficult read but let me reassure you by telling you that we'll go step by step; furthermore, the simple coax geometry will work in our favour.

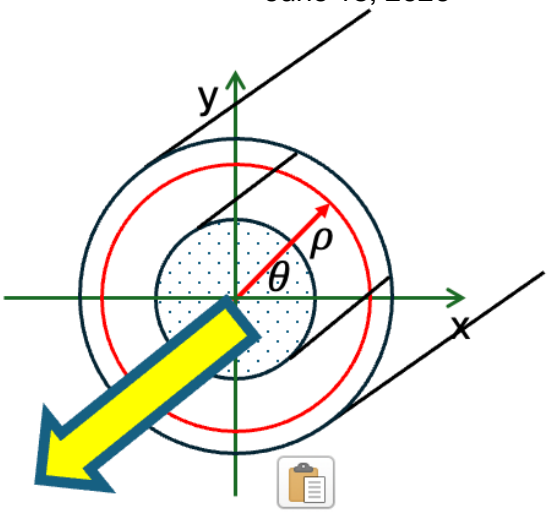
So, let's do the physics! Let's derive the formula for the capacitance and inductance of a coax cable and then get our characteristic impedance... the Maxwell way!

Inductance of a COAX cable

As usual, we start with a physical diagram of our coax cable. This will help our physical intuition and should result in less complicated math.



Coax cross-section



Integration Path

Start with the Maxwell equation for magnetic flux around an enclosed current in the core conductor shown as the yellow arrow.

$$\oint \vec{B} \cdot dl = \mu I$$

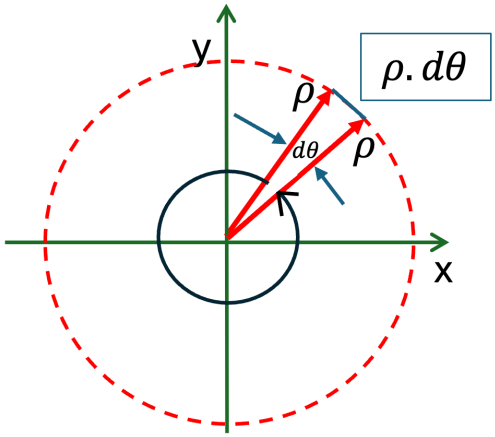
Where,

$\vec{B}$ *is the magnetic flux density* (*measured in Teslas, Gauss, or* $\frac{Webers}{m^2}$)

μ = *permeability of the dielectric material* (*measured in* $\frac{Henry}{m^2}$)

I = *enclosed current* (*measured in Amperes*)

dl = *infinitesimal line elements around core* (*measured in meters*)



Arc length

With ρ as the radius and θ as the angle, we substitute the arc length for the line element, $dl = \rho d\theta$; furthermore, if we add up the small line elements dl around the core of the coax cable we get,

$$\int_0^{2\pi} B_\theta \rho d\theta = \mu I$$

Since the magnetic flux at the constant radius ρ is constant with respect to the angle θ, we can take both terms out from under the integral,

$$B_\theta \rho \int_0^{2\pi} d\theta = \mu I$$

The integral then evaluates to θ at the upper and lower boundaries to be,

$$B_\theta \rho[(2\pi) - (0)] = \mu I$$

$$B_\theta \rho 2\pi = \mu I$$

We'll write B_θ as just B since that's the only component of the magnetic flux.

$$2\pi\rho B = \mu I$$

Solving for B,

$$B = \frac{\mu I}{2\pi\rho} \tag{1}$$

To turn this into an inductance, we look up the energy stored in the field and from that expression we write down the definition of inductance *per unit length* L. (We won't derive *this* expression here and we'll take it as a given.)

$$L = \frac{1}{\mu I^2}\int_a^b B^2 d\tau \tag{2}$$

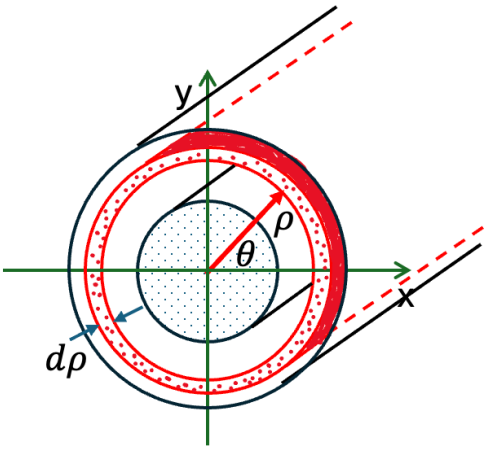
Where τ is the annular volume.

Substitute (1) into (2) and get,

$$L = \frac{1}{\mu I^2}\int_a^b \left(\frac{\mu I}{2\pi\rho}\right)^2 d\tau$$

Simplifying,

$$L = \frac{1}{\cancel{\mu I^2}}\frac{\mu^{\cancel{2}}\cancel{I^2}}{(2\pi)^2}\int_a^b \left(\frac{1}{\rho}\right)^2 d\tau$$



Gauss Volume

Now, $d\tau$ is the volume or thickness of the shell around the coax core. It's thin, but it is a volume. Another way to look at that volume is that it's the perimeter of the shell times the radial element: $d\tau = 2\pi \cdot \rho \cdot d\rho$

We substitute this into our expression and get,

$$L = \frac{\mu}{(2\pi)^2}\int_a^b \left(\frac{1}{\rho}\right)^2 2\pi\rho d\rho$$

Cancelling out some terms,

$$L = \frac{\mu}{(2\pi)^{\cancel{2}}}\int_a^b \left(\frac{1}{\rho}\right)^{\cancel{2}} \cancel{2\pi\rho} d\rho$$

$$L = \frac{\mu}{2\pi}\int_a^b \frac{d\rho}{\rho}$$

Evaluating the integral by looking it up in mathematical tables like the one listed in the references,

$$L = \frac{\mu}{2\pi}[\ln(b) - \ln(a)]$$

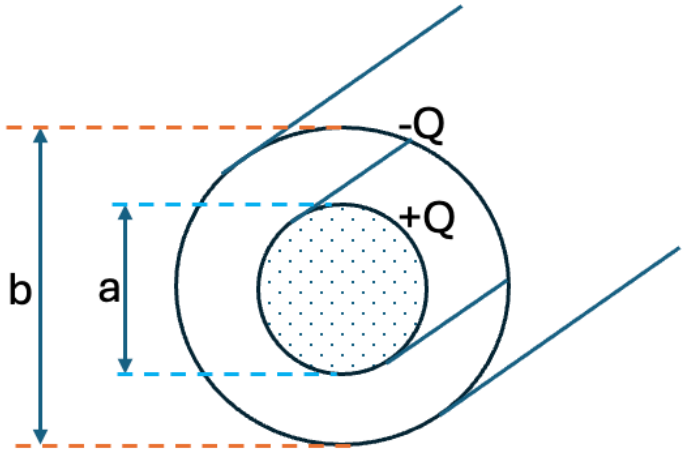
$$L = \frac{\mu}{2\pi}\left[ln\left(\frac{b}{a}\right)\right]$$

Next, we look at the capacitance in the dielectric of our coax cable.

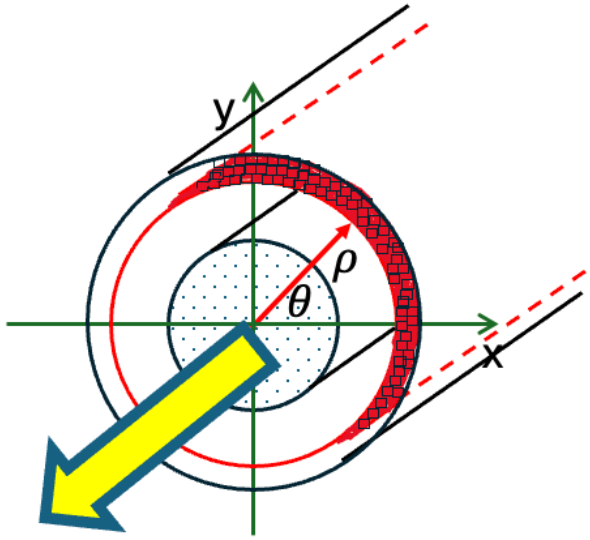
(3)

<u>Capacitance of a COAX cable</u>

As before, we start with a physical diagram of our coax cable.



Coax cross-section



Gauss Surface

Start with the Maxwell equation for the electric field due to a charge Q.

$$\int \vec{E} \cdot ds = \frac{Q}{\epsilon}$$

Where,

$\vec{E}$ *is the electric field* (*measured in* $\frac{V}{m}$)

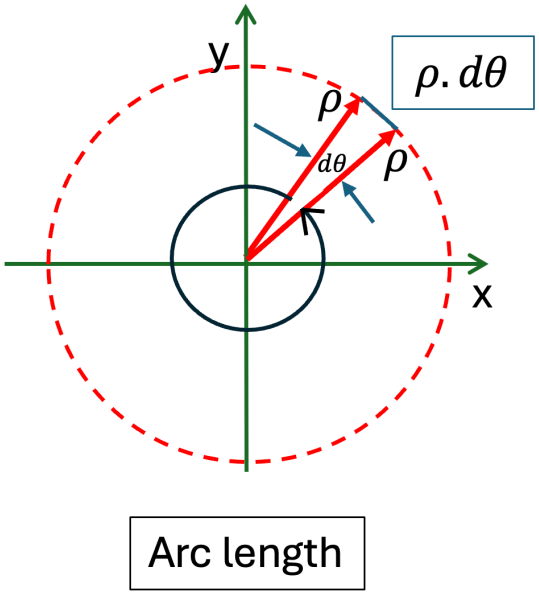
ϵ = *permitivity of dielectric* (*measured in* $\frac{\mathrm{F}}{\mathrm{m}}$)

Q = *enclosed charge per unit length* (*measured in* $\frac{Coulombs}{m}$)

ds = *infinitesimal area elements around core* (*measured in* m^2)

If we substitute the arc length for the surface element, $ds = \rho d\theta dz$ and if we add up the small area elements around the core of the coax cable we get,

$$\int E_\rho \rho d\theta dz = \frac{Q}{\epsilon}$$



We divide both sides by dz and agree to use Q as the *charge per unit length*, which has the surface element's length incorporated in it.

$$\int_0^{2\pi} E_\rho \rho d\theta = \frac{Q}{\epsilon}$$

Next, we replace ϵ for ϵ_o since we are applying this equation to the dielectric portion of the coax cable and not the vacuum. Also, since the E field is constant at the constant radius ρ, we let $E_\rho = E$ and take it out from under the integral sign.

$$E\rho \int_0^{2\pi} d\theta = \frac{Q}{\epsilon}$$

The integral then evaluates to θ at the upper and lower limits,

$$E\rho[(2\pi) - (0)] = \frac{Q}{\epsilon}$$

Which gives,

$$E\rho 2\pi = \frac{Q}{\epsilon}$$

Solving for E,

$$E = \frac{Q}{2\pi\rho\epsilon}$$

(4)

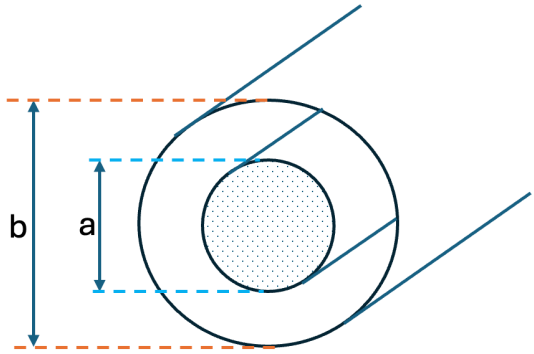
To turn this into a capacitance, we look at the potential difference, or voltage, to bring a test charge from b to a.

$$V = -\int_b^a E\, d\rho$$

(5)

Where ρ is the radial distance from b to a.

Note that we are assuming two perfectly conducting nested cylindrical shapes with the outer one having no thickness. So there is a charge $+Q$ in the core conductor and $-Q$ in the sheath cylinder.



Coax cross-section

Substitute (4) into (5),

$$V = -\int_b^a \frac{Q}{2\pi\rho\epsilon} d\rho$$

Taking the constants out from the under the integral sign,

$$V = -\frac{Q}{2\pi\epsilon}\int_b^a \frac{d\rho}{\rho}$$

Evaluating the integral by looking it up in math tables,

$$V = -\frac{Q}{2\pi\epsilon}[\ln(a) - \ln(b)]$$

Rearranging to absorb the negative sign into the bracketed expression,

$$V = \frac{Q}{2\pi\epsilon}[\ln(b) - \ln(a)]$$

$$V = \frac{Q}{2\pi\epsilon}\left[ln\left(\frac{b}{a}\right)\right]$$

Since the definition of capacitance C is,

$$C = \frac{Q}{V}$$

We substitute and get,

$$C = \frac{\cancel{Q}}{\frac{\cancel{Q}}{2\pi\epsilon}\left[ln\left(\frac{b}{a}\right)\right]}$$

Here we use the definition of the permittivity in terms of the relative permittivity and the permittivity of the vacuum: $\epsilon = \epsilon_o\epsilon_r$

$$C = \frac{2\pi\epsilon_o\epsilon_r}{\left[ln\left(\frac{b}{a}\right)\right]} \quad (6)$$

From previous work we have the characteristic impedance being equal to,

$$Z_o = \sqrt{\frac{L}{C}}$$

Substituting equations (3) and (6),

$$Z_o = \sqrt{\frac{\frac{\mu}{2\pi}\left[ln\left(\frac{b}{a}\right)\right]}{\frac{2\pi\epsilon_o\epsilon_r}{\left[ln\left(\frac{b}{a}\right)\right]}}}$$

$$Z_o = \sqrt{\frac{\mu\left[ln\left(\frac{b}{a}\right)\right]^2}{(2\pi)^2\epsilon_o\epsilon_r}}$$

$$Z_o = \sqrt{\frac{\mu}{\epsilon_o\epsilon_r}}\frac{\left[ln\left(\frac{b}{a}\right)\right]}{2\pi}$$

Here we use the definition of the permeability in terms of the relative permeability and the permeability of the vacuum: $\mu = \mu_o\mu_r$

$$Z_o = \sqrt{\frac{\mu_o\mu_r}{\epsilon_o\epsilon_r}}\frac{\left[ln\left(\frac{b}{a}\right)\right]}{2\pi}$$

and since $\mu_r \cong 1.0$ for coax cables,

$$Z_o = \sqrt{\frac{\mu_o}{\epsilon_o\epsilon_r}}\frac{\left[ln\left(\frac{b}{a}\right)\right]}{2\pi}$$

Plugging and chugging the values for the permeability and permittivity of free space,

$$Z_o = \sqrt{\frac{4\pi\ x\ 10^{-7}}{(8.85\ x\ 10^{-12})\epsilon_r}}\frac{\left[ln\left(\frac{b}{a}\right)\right]}{2\pi}$$

We get,

$$\boxed{Z_o = \frac{59.97}{\sqrt{\epsilon_r}}\left[ln\left(\frac{b}{a}\right)\right]}$$

Per the Belden data sheet in the reference section, for RG8 coax cable, $b = 0.285\ in$ and $a = 0.085\ in$. So, plugging those values in,

$$Z_o = \frac{59.97}{\sqrt{2.1}}\left[ln\left(\frac{0.285}{0.085}\right)\right] = \boxed{50.06\ \Omega}$$

Which is the result that we expected! Woohoo! Physics rocks!

Bibliography

Schaum's Mathematical Handbook of Formulas and Tables, 2nd edition – M.R. Spiegel and J. Liu, McGraw-Hill 1999.

Electromagnetic Fields and Waves, 2nd edition - P. Lorrain and D. Corson, W.H. Freeman and Co. 1970.

Electromagnetism, 2nd edition - I.S. Grant, W.R. Phillips, John Wiley & Sons Ltd. 1990.

Campo Electro-Magnético para físicos e Ingenieros - Rafael Gómez Martín, Editorial Universidad de Granada 2021.

Zo of the RG-8X cable Part 3 – YouTube.com @QuantumRadio
https://youtu.be/PkMvS1v4eKE?si=0Gmv6k2_hOPpsDRX

Belden data sheet for product no. 8237 - 50 Ohm Wireless Transmission Coax, RG-8, 13 AWG Str BC, 95% BC Braid, PVC Jkt, CMH
https://catalog.belden.com/techdata/EN/8237_techdata.pdf